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Exercises for 'Functional Analysis 2' [MATH-404]

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Ex 8.1 (Distributional derivatives as difference quotients)
Let $T \in \mathcal{D}'(\mathbb{R}^d)$ be a distribution. For $h > 0$ and $1 \leq j \leq d$ define $\tau_{h,i} : \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}(\mathbb{R}^d)$ by $(\tau_h \varphi)(x) = \varphi(x - he_i)$.

a) Justify why $(\tau_{h,i}T)(\varphi) := T(\tau_{-h,i}\varphi)$ defines a distribution.

b) Show that
$$\lim_{h \to 0} \frac{T - \tau_{h,i}T}{h} = D^{e_i}T \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$
In this sense, the distributional derivatives are still limits of difference quotients.

Solution 8.1 : a) Clearly $\tau_{h,i}T$ is linear in φ. By Exercise 6.2-a)-iii), the operation $\tau_{-h,i}$ is continuous from $\mathcal{D}(\mathbb{R}^d)$ to itself. Hence the composition with T defines continuous functional on $\mathcal{D}(\mathbb{R}^d)$.

b) Fix $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and denote its support by K. Then $\tau_{-h,i}\varphi$ has support in $K - he_i$; therefore we can find a compact set $\widetilde{K}$ such that $\text{supp}(\varphi - \tau_{-h,i}\varphi) \subset \widetilde{K}$ uniformly over all $|h| \leq 1$. Moreover, for any multi-index α, a second order Taylor expansion of $D^\alpha \varphi$ along the line $[x, x + he_i]$ yields that
$$\begin{aligned}\left| D^\alpha \left(\frac{1}{h} \left(\varphi(x) - \tau_{-h,i}\varphi(x) + D^{e_i}\varphi(x)h \right) \right) \right| =& \frac{1}{h} \left| (-D^\alpha \varphi(x) + (D^\alpha \varphi)(x + he_i) - D^{e_i} D^\alpha \varphi(x)h \right| \\ \leq& \frac{1}{2} \sup_{x \in \mathbb{R}^d} |D^{\alpha + 2e_i}\varphi(x)| h \to 0 \quad \text{ as } h \to 0.\end{aligned}$$

Hence we proved that
$$\frac{1}{h} \left(\varphi - \tau_{-h,i}\varphi \right) \to -D^{e_i}\varphi \text{ in } \mathcal{D}(\Omega).$$
We deduce that
$$\frac{T - \tau_{h,i}T}{h}(\varphi) = T\left(\frac{1}{h} \left(\varphi - \tau_{-h,i}\varphi \right) \right) \overset{h \to 0}{\longrightarrow} T(-D^{e_i}\varphi) = D^{e_i}T(\varphi).$$

Ex 8.2 (Fourier transform and distributional derivatives*)
Let $\alpha \in \mathbb{N}_0^d$ be a multi-index.

a) Show that the function $\mathbb{R}^d \to \mathbb{R} : \ x \mapsto x^\alpha$ is a tempered distribution.

b) Prove that for any $T \in \mathscr{S}'(\mathbb{R}^d)$
$$\widehat{D^\alpha T} = (ik)^\alpha \, \widehat{T} \quad \text{and} \quad \widehat{x^\alpha \, T} = (iD)^\alpha \widehat{T}.$$

Hint: Use the corresponding identities for the Fourier transform on $\mathscr{S}(\mathbb{R}^d)$.

c) Show that $\widehat{\delta_0} = 1$ and $\hat{1} = (2\pi)^d \delta_0$. Then demonstrate the following identities in $\mathscr{S}'(\mathbb{R}^d)$

$$\widehat{D^\alpha \delta_0} = (ik)^\alpha \quad \text{and} \quad \widehat{x^\alpha} = (2\pi)^d (iD)^\alpha \delta_0.$$

Ex 8.3 (Fourier transform of p.v.$(1/x)$)
Let

$$T = \text{p.v.}\left(\frac{1}{x}\right).$$

a) Show that T is a tempered distribution on $\mathbb{R}$.
Hint: Use the formula for T derived in the solution to Ex. 7.1.

b) Show that $\widehat{T}$ is a solution of the differential equation in $\mathscr{S}'(\mathbb{R})$

$$iD\widehat{T} = 2\pi\delta_0.$$

Hint: Start with the identity $x \cdot T = 1$.

c) Use b) to compute that $\widehat{T} = -i\pi\,\text{sign}$, where sign is the signum function

$$\text{sign}(x) = 1 \quad (x > 0), \quad \text{sign}(x) = -1 \quad (x < 0), \quad \text{sign}(x) = 0 \quad (x = 0).$$

Hint: Employ Ex 7.3 and the fact that T is an **odd distribution**, i.e., $T(\varphi(-\cdot)) = -T(\varphi)$.

Solution 8.3 :
a) Recall that

$$\text{p.v.}\left(\frac{1}{x}\right)(\varphi) = \int_0^\infty \frac{\varphi(x) - \varphi(-x)}{x}\,dx.$$

As in the solution of Exercise 7.1 we can prove that

$$\left|\frac{\varphi(x) - \varphi(-x)}{x}\right| \le 2 \sup_{y\in\mathbb{R}} |\varphi'(y)|.$$

Hence we can estimate

$$\begin{aligned}\left|\text{p.v.}\left(\frac{1}{x}\right)\varphi(x)\right| =& \int_0^1 2\sup_{y\in\mathbb{R}} |\varphi'(y)| dx + \int_{\{|x|\ge 1\}} |x|^2 |\varphi(x)| \cdot |x|^{-2} dx \\ \le& 2\sup_{y\in\mathbb{R}} |\varphi'(y)| + 2\sup_{y\in\mathbb{R}} |y|^2 |\varphi(y)| \\ \le& 2(p_{0,1}(\varphi) + p_{2,0}(\varphi))\end{aligned}$$

with the seminorms given by Definition 2.26.

b) Applying the Fourier transform to both sides of the identity $x \cdot T = 1$ we get from Ex 8.2 b) and c) that

$$iD\widehat{T} = \hat{1} = 2\pi\delta_0.$$

c) Observe that sign $= 2H - 1$, where $H = \mathbb{1}_{[0,+\infty)}$ is the Heaviside function ; by Remark 2.22, we know that $\delta_0 = DH$. Thus $D(\text{sign}) = 2\delta_0$ and so we can rewrite b) as

$$D(i\widehat{T} - \pi\text{sign}) = 0.$$

Thus, from Ex 7.3-b) there exists a constant $c \in \mathbb{C}$ such that [1]

$$\widehat{T} = -i\pi \text{sign} - ic.$$

It remains to show that $c = 0$. Notice that, since T is odd, so is $\widehat{T}$: by a change of variables, one can show that $\mathcal{F}[\varphi(-\cdot)])(k) = \mathcal{F}[\varphi](-k))$. Since $\hat{T}$ is odd, sign is odd (check it by definition), but c is even, it must necessarily hold that $c = 0$. Indeed, for any $\varphi \in \mathscr{S}(\mathbb{R}^d)$ it holds that

$$\begin{aligned} i\pi\text{sign}(\varphi) + ic(\varphi) = -\widehat{T}(\varphi) = \widehat{T}(\varphi(-\cdot)) &= -\, i\pi\text{sign}(\varphi(-\cdot)) - ic(\varphi(-\cdot)) \\ &= i\pi\text{sign}(\varphi) - ic(\varphi) \end{aligned}$$

from which it follows that $c = 0$.

Ex 8.4 (Two applications of the fundamental lemma of the calculus of variations)
a) For an open set $\Omega \subset \mathbb{R}^d$, we define the so-called **Sobolev space**

$$W^{1,p}(\Omega) = \{f \in L^p(\Omega) :\ D^{e_i} f \in L^p(\Omega)\ \forall i = 1, \dots, d\},$$

where D^{e_i} denotes the distributional derivative. In this case we say that f is **weakly differentiable** and $D^{e_i} f$ is the weak i-th partial derivative. Show that the weak derivative is unique.

b) Let $\Omega \subset \mathbb{R}^d$ be open and $1 \le p < +\infty$. Show that $\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$.
Hint: Recall that by the Riesz representation theorem, the dual space of $L^p(\Omega)$ can be identified with $L^q(\Omega)$ with $p^{-1} + q^{-1} = 1$.

Solution 8.4 : a) Requiring that the distributional derivative belongs to $L^p(\Omega)$ means that there exists $g \in L^p(\Omega)$ such that

$$\int_\Omega \varphi(x) g(x)\,\mathrm{d}x = -\int_\Omega D^{e_i}\varphi(x) f(x)\,\mathrm{d}x$$

for all $\varphi \in \mathcal{D}(\Omega)$. If there are two such functions $g_1, g_2 \in L^p(\Omega)$, then their difference satisfies

$$\int_\Omega \varphi(x)(g_1(x) - g_2(x))\,\mathrm{d}x = 0$$

for all $\varphi \in \mathcal{D}(\Omega)$. Hence the fundamental lemma of the calculus of variations implies that $g_1 = g_2$ a.e. which proves uniqueness as elements in $L^p(\Omega)$.

b) Assume by contradiction that the L^p-closure of $\mathcal{D}(\Omega)$, denoted henceforth by X, is a proper subset of $L^p(\Omega)$. Hence there exists $f \in L^p(\Omega) \setminus X$. Since the set $\{f\}$ is compact and X is closed, and both sets are convex [2], by the geometric version of the Hahn-Banach theorem (for real vector spaces) there exists $G \in L^p(\Omega)'$ such that

$$G(f) < G(\varphi) \quad \forall \varphi \in X. \tag{1}$$

By the Riesz representation theorem we know that $L^p(\Omega)' = L^q(\Omega)$ [3] with $p^{-1} + q^{-1} = 1$, so that there exists $g \in L^q(\Omega) \subset L^1_{\text{loc}}(\Omega)$ such that

$$\int_\Omega g(x) f(x)\,\mathrm{d}x < \int_\Omega g(x)\varphi(x)\,\mathrm{d}x$$

1. Exercise 7.3 was stated for distributions over $\mathbb{R}$, so the conclusion was with $c \in \mathbb{R}$. By separating real and imaginary parts of the complex-valued distribution $\hat{T}$, here one can achieve the same conclusion with $c \in \mathbb{C}$.

2. In abstract topological vector spaces, one can show that the closure of a convex set is always convex; the most direct way to prove it is using nets (which you are not required to know). Here, since $L^p(\Omega)$ is a Banach space (thus metrizable), closed sets are equivalent to sequentially closed ones, so it suffices to work with sequences (which instead you should know how to manipulate).

3. This is the key step where the restriction $p < \infty$ is needed. Indeed the statement is not true for $p = \infty$.

for all $\varphi \in \mathcal{D}(\Omega)$. Since the left-hand side is fixed, we can vary φ by $\lambda\varphi$ for all $\lambda \in \mathbb{R}$ and deduce that

$$\int_\Omega g(x)\varphi(x)\,\mathrm{d}x = 0$$

for all $\varphi \in \mathcal{D}(\Omega)$. The fundamental lemma of the calculus of variations yields that $g = 0$ a.e. which yields a contradiction to (1).